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Quantum mechanics on the h -deformed quantum plane

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Abstract. We find the covariant deformed Heisenberg algebra and the Laplace–Beltrami operator on the extended h -deformed quantum plane and solve the Schrödinger equations explicitly for some physical systems on the quantum plane. In the commutative limit the behaviour of a quantum particle on the quantum plane becomes that of the quantum particle on the Poincaré half-plane, a surface of constant negative Gaussian curvature. We show that the bound state energy spectra for particles under specific potentials depend explicitly on the deformation parameter h . Moreover, it is shown that bound states can survive on the quantum plane in a limiting case where bound states on the Poincaré half-plane disappear.

1. Introduction

In recent years, quantum groups and quantum spaces have attracted much attention from both physics and mathematics [6, 15, 18]. They are of relevance to the problem of the quantization of spacetime. Quantum spaces are often adopted as models for the microscopic structure of physical spacetime and their effects on physics have been investigated by many authors. In particular, q -deformed quantum spaces are one of those quantum spaces whose effects have been intensively studied (see, e.g., [2, 8, 13]).

The purpose of this work is to make a formulation of quantum mechanics on the h -deformed quantum plane and to investigate the effects of the quantum plane on the energy spectra. The h -deformed quantum plane [1, 7, 12] is a counterpart of the q -deformed one in the set of quantum planes which are covariant under those quantum deformations of $GL(2)$ which admit a central determinant. It seems that it has more geometrical structures than the q -deformed one. The h -deformed quantum plane is known to be a noncommutative version of the Poincaré half-plane and its geometry has been discussed in [5].

In section 2, we review the geometry of the h -deformed quantum plane which is to be used later. In section 3, the h -deformed Heisenberg algebra is constructed from the skew derivatives of Wess–Zumino. This algebra is covariant under the quantum group $GL_h(2)$. It is worthy of being compared with the deformed Heisenberg algebra used by Aghamohammadi [1], which is not covariant under the quantum group. Also it is comparable with the deformed Heisenberg algebra, for example, by Lorek *et al* [13], which has no such quantum group symmetry because there is none compatible with the reality condition. We also construct the Laplace–Beltrami operator on the quantum plane. The operator has the Laplace–Beltrami operator of the Poincaré half-plane as its commutative limit. In section 4, we construct the Schrödinger equations for some physical systems on the quantum plane and find their solutions explicitly, taking values in the noncommutative algebra. In the commutative limit the behaviour of a quantum particle on the quantum plane becomes that of the quantum particle

on the Poincaré half-plane, a surface of constant negative Gaussian curvature. The bound state energy spectra for particles under specific potentials are shown to depend explicitly on the deformation parameter h . This result is comparable with that of [3]. Moreover, it is shown that bound states can survive on the quantum plane in a limiting case where bound states on the Poincaré half-plane disappear.

2. The h -deformed quantum plane

2.1. The covariant differential structure

The h -deformed quantum plane is an associative algebra generated by noncommuting coordinates x and y such that

$$xy - yx = hy^2 \quad (2.1.1)$$

where h is a deformation parameter. The quantum group $GL_h(2)$ is the symmetry group of the h -deformed plane as is $GL_q(2)$ for the q -deformed quantum plane [4, 7, 12].

The covariant differential calculus on the quantum plane can be found [1] by the method of Wess and Zumino [21]. The results to be used in this work can be summarized as follows. The module structure of the 1-forms is given by the relations

$$\begin{aligned} x dx &= dx x - h dx y + h dy x + h^2 dy y & x dy &= dy x + h dy y \\ y dx &= dx y - h dy y & y dy &= dy y \end{aligned} \quad (2.1.2)$$

and the structure of the algebra of forms is determined by the relations

$$dx^2 = h dx dy \quad dx dy = -dy dx \quad dy^2 = 0. \quad (2.1.3)$$

It is important to notice that the associative algebra generated by x, y has an involution given by $x^\dagger = x, y^\dagger = y$ provided $h \in i\mathbb{R}$. The involution has a natural and simple extension to the differential calculus and the differential is real:

$$(df)^\dagger = df^\dagger. \quad (2.1.4)$$

This is contrary to the case considered by Lorek *et al* [13]. Dual to these forms are a set of twisted derivations which when considered as operators satisfy the relations

$$\begin{aligned} [x, \partial_x] &= -1 + hy \partial_x & [x, \partial_y] &= -hx \partial_x - h^2 y \partial_x - hy \partial_y \\ [y, \partial_x] &= 0 & [y, \partial_y] &= -1 + hy \partial_x \\ [\partial_x, \partial_y] &= h \partial_x^2. \end{aligned} \quad (2.1.5)$$

In [1] the momentum operators are defined as follows:

$$p_x = -i\hbar \partial_x \quad p_y = -i\hbar \partial_y. \quad (2.1.6)$$

However, if one requires that they be Hermitian, the deformed Heisenberg algebra given by (2.1.5) does not satisfy the hermiticity. This problem of nonhermiticity has been observed by Aghamohammadi [1]. We shall resolve it in subsection 3.1.

The extended h -deformed quantum plane is an associative algebra \mathcal{A} generated by x, y satisfying equation (2.1.1) and their inverses x^{-1}, y^{-1} . The extended h -deformed quantum plane is known to be a noncommutative version of a Poincaré half-plane [5]. Since \mathcal{A} is a unital involution algebra with x and y Hermitian and $h \in i\mathbb{R}$, the elements

$$u = xy^{-1} + \frac{1}{2}h \quad v = y^{-2} \quad (2.1.7)$$

are also Hermitian. Their commutation relation becomes

$$[u, v] = -2hv. \quad (2.1.8)$$

This choice of generators is useful in studying the commutative limit.

Now it is straightforward to see that

$$\begin{aligned} u du &= du u - 2h du & u dv &= dv u - 2h dv \\ v du &= du v & v dv &= dv v \end{aligned} \quad (2.1.9)$$

and

$$du du = dv dv = 0 \quad du dv = -dv du. \quad (2.1.10)$$

In this work, however, it is more convenient to use the ‘Stehbein’ [16].

2.2. The Stehbein

The Stehbein θ^a are defined as

$$\theta^1 = v^{-1} du \quad \theta^2 = -v^{-1} dv. \quad (2.2.1)$$

Then the θ^a satisfy the commutation relations

$$f\theta^a = \theta^a f \quad (2.2.2)$$

for any $f \in \mathcal{A}$ as well as the relations

$$(\theta^1)^2 = 0 \quad (\theta^2)^2 = 0 \quad \theta^1\theta^2 + \theta^2\theta^1 = 0. \quad (2.2.3)$$

Also we have

$$d\theta^1 = -\theta^1\theta^2 \quad d\theta^2 = 0. \quad (2.2.4)$$

Moreover, the Stehbein satisfy

$$\theta^a(e_b) = \delta_b^a \quad (2.2.5)$$

if we introduce the derivations $e_a = \text{ad } \lambda_a$ with

$$\lambda_1 = \frac{1}{2h}v \quad \lambda_2 = \frac{1}{2h}u. \quad (2.2.6)$$

It is easy to see that the derivations satisfy

$$\begin{aligned} e_1u &= v & e_1v &= 0 \\ e_2u &= 0 & e_2v &= -v. \end{aligned} \quad (2.2.7)$$

The derivations e_a define, in the commutative limit, vector fields

$$X_a = \lim_{h \rightarrow 0} e_a \quad (2.2.8)$$

with

$$X_1 = \tilde{v} \partial_{\tilde{u}} \quad X_2 = -\tilde{v} \partial_{\tilde{v}} \quad (2.2.9)$$

where \tilde{u}, \tilde{v} are the commutative limits of the generators u, v of the algebra \mathcal{A} . The algebra \mathcal{A} with the differential calculus defined by the relations (2.2.3) can be regarded as a noncommutative deformation of the Poincaré half-plane [5]. In this case, a metric is defined on the tensor product of the \mathcal{A} -module of 1-forms by

$$g(\theta^a \otimes \theta^b) = \delta^{ab}. \quad (2.2.10)$$

The metric satisfies

$$g(du \otimes du) = g(dv \otimes dv) = v^2 \quad g(du \otimes dv) = g(dv \otimes du) = 0. \quad (2.2.11)$$

In terms of the commutative limit $\tilde{\theta}^a$ of the Stehbein θ^a , the metric is given in the commutative limit by the line element

$$ds^2 = (\tilde{\theta}^1)^2 + (\tilde{\theta}^2)^2 = \tilde{v}^{-2}(d\tilde{u}^2 + d\tilde{v}^2) \quad (2.2.12)$$

which is the metric of the Poincaré half-plane [10].

3. The h -deformed Heisenberg algebra and the Laplace–Beltrami operator

3.1. The h -deformed Heisenberg algebra

In this subsection, we shall construct a deformed Heisenberg algebra on the extended h -deformed quantum plane. First, we shall introduce ∂_u, ∂_v such that

$$\partial_u u = \partial_v v = 1. \quad (3.1.1)$$

We assume the following ansatz:

$$\partial_x = (\partial_x u) \partial_u + (\partial_x v) \partial_v \quad \partial_y = (\partial_y u) \partial_u + (\partial_y v) \partial_v. \quad (3.1.2)$$

The coefficients can be calculated using equation (2.1.5),

$$\begin{aligned} \partial_x u &= y^{-1} & \partial_x v &= 0 \\ \partial_y u &= -xy^{-2} - hy^{-1} & \partial_y v &= -2y^{-3} \end{aligned} \quad (3.1.3)$$

and thus we have

$$\partial_u = y\partial_x \quad \partial_v = -\frac{1}{2}(xy^2 - 2hy^3)\partial_x - \frac{1}{2}y^3\partial_y. \quad (3.1.4)$$

Then ∂_u, ∂_v satisfy not only equation (3.1.1) but also

$$\partial_u v = \partial_v u = 0. \quad (3.1.5)$$

Now that

$$\begin{aligned} \partial_u x &= y & \partial_u y &= 0 \\ \partial_v x &= -\frac{1}{2}(xy^2 - 2hy^3) & \partial_v y &= -\frac{1}{2}y^3 \end{aligned} \quad (3.1.6)$$

we can verify the relations

$$\partial_u = (\partial_u x) \partial_x + (\partial_u y) \partial_y \quad \partial_v = (\partial_v x) \partial_x + (\partial_v y) \partial_y. \quad (3.1.7)$$

Moreover, it is straightforward to see that

$$d = dx \partial_x + dy \partial_y = du \partial_u + dv \partial_v. \quad (3.1.8)$$

From equations (2.1.5) and (3.1.7), it follows that

$$\begin{aligned} [u, \partial_u] &= -1 + 2h\partial_u & [u, \partial_v] &= 2h\partial_v \\ [v, \partial_u] &= 0 & [v, \partial_v] &= -1 \\ [\partial_u, \partial_v] &= 0. \end{aligned} \quad (3.1.9)$$

Now we introduce the momentum operators temporarily by

$$p_u = -i\hbar\partial_u \quad p_v = -i\hbar\partial_v \quad (3.1.10)$$

then we have the following commutation relations:

$$\begin{aligned} [u, p_u] &= i\hbar + 2hp_u & [u, p_v] &= 2hp_v \\ [v, p_u] &= 0 & [v, p_v] &= i\hbar \\ [p_u, p_v] &= 0. \end{aligned} \quad (3.1.11)$$

These relations satisfy the Jacobi identities. It is worth noticing that the commutation relations are invariant under the transformation

$$p'_u = p_u \quad p'_v = p_v + \alpha v^{-1} \quad (3.1.12)$$

for any complex number α . Moreover, equation (3.1.11) satisfies the hermiticity if we define the Hermitian adjoints of the momentum operators as follows:

$$p_u^\dagger = p_u \quad p_v^\dagger = p_v + \beta v^{-1} \tag{3.1.13}$$

for any complex number β .

From now on, we choose $\beta = 2i\hbar$ in equation (3.1.13). In fact, not only does this choice guarantee the hermiticity of the Laplace–Beltrami operator to be constructed in the next subsection, but also it is consistent with that in the commutative limit [10, 17] once one defines the inner product $\langle f, g \rangle$ of two functions on the Poincaré half-plane to be $\langle f, g \rangle = \int \bar{f}g \, d\mu$ with the measure $d\mu = \tilde{v}^{-2} \, d\tilde{u} \, d\tilde{v}$. It follows then that $p_{\tilde{v}}^\dagger = p_{\tilde{v}} + 2i\hbar \tilde{v}^{-1}$ and the Hermitian momentum operator is not $p_{\tilde{v}}$ but $P_{\tilde{v}} \equiv p_{\tilde{v}} + i\hbar \tilde{v}^{-1}$. Similarly, the momentum operator is not p_v but $P_v \equiv p_v + i\hbar v^{-1}$ in the quantum plane. This is not unusual either in nonflat spaces [11] or in quantum spaces [8]. In fact, the second term of P_v reflects the nonflatness of the space. From equation (3.1.12), it follows that the Hermitian momentum operators $P_u \equiv p_u$ and P_v satisfy the following h -deformed Heisenberg algebra on the (u, v) -quantum plane which is of the same form as in equation (3.1.11):

$$\begin{aligned} [u, P_u] &= i\hbar + 2h P_u & [u, P_v] &= 2h P_v \\ [v, P_u] &= 0 & [v, P_v] &= i\hbar \\ [P_u, P_v] &= 0. \end{aligned} \tag{3.1.14}$$

In this case, the Hermitian adjoints of p_x and p_y in equation (2.1.6) are given by

$$p_x^\dagger = p_x \quad p_y^\dagger = p_y + 2hp_x \tag{3.1.15}$$

and the operator such as p_y is not Hermitian, in contrast to the assumption in [1]. In fact, the Hermitian momentum operators are $P_x \equiv p_x$ and $P_y \equiv p_y + hp_x$. Now we can write equation (2.1.5) as

$$\begin{aligned} [x, p_x] &= i\hbar + hyp_x & [x, p_y] &= -hxp_x - h^2yp_x - hyp_y \\ [y, p_x] &= 0 & [y, p_y] &= i\hbar + hyp_x \\ [p_x, p_y] &= hp_x^2. \end{aligned} \tag{3.1.16}$$

It is straightforward to see that the relations in the above equation (3.1.16) satisfy the hermiticity. Moreover, a lengthy calculation shows that they are covariant under the quantum group $GL_h(2)$. Now equation (3.1.16) yields immediately the h -deformed Heisenberg algebra on the (x, y) -quantum plane

$$\begin{aligned} [x, P_x] &= i\hbar + hyP_x & [x, P_y] &= i\hbar h - hxP_x + h^2yP_x - hyP_y \\ [y, P_x] &= 0 & [y, P_y] &= i\hbar + hyP_x \\ [P_x, P_y] &= hP_x^2. \end{aligned} \tag{3.1.17}$$

From the covariance of equation (3.1.16) it follows easily that this deformed Heisenberg algebra is also covariant under the quantum group $GL_h(2)$. The deformed Heisenberg algebra is comparable with that in [1] which is not covariant under the quantum group $GL_h(2)$.

3.2. The Laplace–Beltrami operator

From now on, we shall concern the (u, v) -plane. From equation (2.2.3), the differential algebra over \mathcal{A} is the sum of \mathcal{A} and the \mathcal{A} -modules Ω^1, Ω^2 of 1-forms and 2-forms:

$$\Omega(\mathcal{A}) = \mathcal{A} \oplus \Omega^1 \oplus \Omega^2. \tag{3.2.1}$$

As in the classical geometry, we can define the star operator $*$ on $\Omega(\mathcal{A})$ with the metric given in equation (2.2.10) by

$$\begin{aligned} *1 &= \theta^1\theta^2 & *\theta^1\theta^2 &= 1 \\ *\theta^1 &= \theta^2 & *\theta^2 &= -\theta^1. \end{aligned} \quad (3.2.2)$$

Then for $\theta \equiv -\lambda_a\theta^a$, we have $*\theta = -\lambda_1\theta^2 + \lambda_2\theta^1$ and as in classical geometry the star operator $*$ satisfies

$$** = (-1)^{p(2-p)} \quad (3.2.3)$$

where p denotes the order of the form to be acted on. Now we define $\delta : \Omega^p \rightarrow \Omega^{p-1}$ by

$$\delta\varpi = (-1)^{2(p+1)+1} * d * \varpi. \quad (3.2.4)$$

From equation (2.2.4), it follows then that for $f \in \mathcal{A}$

$$\begin{aligned} \delta f &= 0 & \delta(\theta^1\theta^2) &= 0 \\ \delta\theta^1 &= 0 & \delta\theta^2 &= -1. \end{aligned} \quad (3.2.5)$$

Moreover, a straightforward calculation yields

$$\begin{aligned} \delta(f\theta^1\theta^2) &= e_2f\theta^1 - e_1f\theta^2 & \delta\theta &= \lambda_2 \\ \delta(f\theta) &= (e_1f)\lambda_1 + (e_2f)\lambda_2 + f\lambda_2 & \delta(\theta f) &= \lambda_1e_1f + \lambda_2e_2f + \lambda_2f. \end{aligned} \quad (3.2.6)$$

If we define the Laplace–Beltrami operator Δ to be

$$-\Delta = \delta d + d\delta \quad (3.2.7)$$

then it is easy to see that for any $f \in \mathcal{A}$

$$\Delta f = e_1^2 f + e_2^2 f + e_2 f. \quad (3.2.8)$$

From equation (2.2.9), in the commutative limit, it follows that Δ goes over to

$$\tilde{\Delta} \equiv \tilde{v}^2(\partial_u^2 + \partial_v^2) \quad (3.2.9)$$

which is the Laplace–Beltrami operator on the Poincaré half-plane [10]. Thus this result is consistent with that of [5] that the extended \hbar -deformed quantum plane is a noncommutative version of the Poincaré half-plane, a surface of constant negative Gaussian curvature.

Now we claim that the operators e_1 and e_2 are nothing but $v\partial_u$ and $-v\partial_v$, respectively. In fact, the first claim can be seen from mathematical induction together with the following observations:

$$\begin{aligned} e_1u &= v\partial_uu & e_1u^{-1} &= v\partial_uu^{-1} \\ e_1v &= v\partial_uv & e_1v^{-1} &= v\partial_uv^{-1} \end{aligned} \quad (3.2.10)$$

and

$$\begin{aligned} e_1(uf) &= v\partial_u(uf) & e_1(u^{-1}f) &= v\partial_u(u^{-1}f) \\ e_1(vf) &= v\partial_u(vf) & e_1(v^{-1}f) &= v\partial_u(v^{-1}f) \end{aligned} \quad (3.2.11)$$

where we have used the identity that $\partial_u f = u^{-1}f + u\partial_u(u^{-1}f) - 2\hbar\partial_u(u^{-1}f)$. Similarly, the second claim can also be proved.

Moreover, from the above claims it follows that

$$\Delta = v^2(\partial_u^2 + \partial_v^2) = -\frac{1}{\hbar^2}v^2(p_u^2 + p_v^2). \quad (3.2.12)$$

It is straightforward to see that Δ is Hermitian for the choice of $\beta = 2i\hbar$ in equation (3.1.13). Moreover, the Laplace–Beltrami operator Δ can be written as

$$\Delta = \frac{1}{\sqrt{|g|}} \partial_\mu \sqrt{|g|} g^{\mu\nu} \partial_\nu \quad (3.2.13)$$

as expected, where $g^{\mu\nu} = g(du^\mu \otimes du^\nu)$ for $u^\mu = (u, v)$ and $|g| = \det(g_{\mu\nu})$.

4. Quantum mechanics on the quantum plane

4.1. Free particles

Now we shall discuss quantum mechanics on the h -deformed quantum plane. We assume [14, 17] that the Schrödinger equation of a particle with mass m in a potential V is given by

$$i\hbar \partial_t \Psi = \hat{H} \Psi \tag{4.1.1}$$

where $\hat{H} = -(\hbar^2/2m)\Delta + V(u, v)$ and time t is regarded as an extra commuting coordinate. As in the commutative case, it is enough to find the solutions of the time-independent Schrödinger equation.

First let us solve the time-independent Schrödinger equation of a free particle of mass m

$$-\frac{\hbar^2}{2m} \Delta \Psi(u, v) = E \Psi(u, v). \tag{4.1.2}$$

Let us put $\Psi(u, v) = g(v) f(u)$ and $\lambda = -2mE/\hbar^2$. Then we have

$$v^2 (\partial_u^2 + \partial_v^2) g(v) f(u) = \lambda g(v) f(u). \tag{4.1.3}$$

We can decompose this equation into two equations without difficulty using the relations in equations (3.1.5) and (3.1.9)

$$\partial_u^2 f(u) = -C^2 f(u) \tag{4.1.4}$$

and

$$v^2 \partial_v^2 g(v) = (C^2 v^2 + \lambda) g(v) \tag{4.1.5}$$

where C is a constant. In the case when $C = 0$, it follows that up to constant multiplication $f = 1$ and for any number $\alpha \neq 0$

$$g(v) = v^\alpha \tag{4.1.6}$$

with $E = -\hbar^2 \alpha(\alpha - 1)/2m$. However, this solution is not normalizable in the commutative limit. Thus we suppose that $C \neq 0$. From the commutation relations in equation (3.1.9), it follows that for any constant a

$$\partial_u e^{au} = \frac{1 - e^{-2ah}}{2h} e^{au} \tag{4.1.7}$$

where $e^{au} = 1 + au + \frac{1}{2!}(au)^2 + \dots$ is a formal power series. Thus the solution of the equation in (4.1.4) is given by

$$f(u) = e^{iku} \tag{4.1.8}$$

with

$$C^2 = \left[\frac{1 - e^{-2ikh}}{2|h|} \right]^2 \tag{4.1.9}$$

which goes over to k^2 in the commutative limit $h \rightarrow 0$. On the other hand, since the commutation relations between v, ∂_v and $\tilde{v}, \partial_{\tilde{v}}$ are of the same form, we can regard equation (4.1.5) as an ordinary differential equation although the solutions of the quantum plane are formal. Let $\lambda = -\kappa^2 - \frac{1}{4}$. Now, if we put $z = iCv$ and $g(v) = \sqrt{z}\phi(z)$, then the equation (4.1.5) becomes the Bessel differential equation

$$\phi''(z) + \frac{1}{z}\phi'(z) + \left(1 + \frac{\kappa^2}{z^2}\right)\phi(z) = 0. \tag{4.1.10}$$

Thus the solution of the differential equation in (4.1.5) is given by

$$g(v) = \sqrt{v} K_{i\kappa}(|C|v). \quad (4.1.11)$$

The energy eigenvalues $E = -(\hbar^2/2m)\lambda = (\hbar^2/2m)(\kappa^2 + \frac{1}{4})$ for $\kappa > 0$ constitute a continuous spectrum. The largest lower bound state with $\kappa = 0$ is not allowed since the normalized wavefunction vanishes identically in the commutative limit [10]. In the limit the Green function is known to have a cut on the positive real axis in the complex energy plane with a branch point at $E = \hbar^2/8m$. The Bessel functions in these solutions seem to be related to the ‘cylindrical’ topology of the Poincaré half-plane [20].

For the classical mechanics, the equations of motion with the Hamiltonian

$$H = \frac{\tilde{v}^2}{2m} (P_u^2 + P_v^2) \quad (4.1.12)$$

yield the geodesic equation on the Poincaré half-plane:

$$\ddot{u} = \frac{2\dot{u}\dot{v}}{\tilde{v}} \quad \ddot{v} = \frac{1}{\tilde{v}} (\dot{v}^2 - \dot{u}^2). \quad (4.1.13)$$

We shall conclude this subsection with the following observation. As in the q -deformed case (see, e.g., [19]), the first relation in equation (3.1.9) is the one whose Leibnitz rule does not involve the coordinate v . Thus let us assume a one-dimensional h -deformed Heisenberg relation in the u -direction from equation (3.1.14) as follows:

$$[u, P_u] = i\hbar + 2h P_u \quad (4.1.14)$$

and choose the Hamiltonian as $\hat{H} = (P_u^2)/2m$. Then the one-dimensional time-dependent Schrödinger equation has a solution of the form

$$\Psi(t, u) = e^{i(ku - \omega t)} \quad (4.1.15)$$

provided with

$$\omega = \frac{\hbar}{2m} \left[\frac{1 - e^{-2ikh}}{2|h|} \right]^2. \quad (4.1.16)$$

Thus the energy depends on the parameter h explicitly, which is similar to the one-dimensional q -deformed case [3]. However, the h dependence does not arise in this way for a free particle on the h -deformed quantum plane. We investigate the h dependence of the energy spectra for some bound states in the following subsections.

4.2. Motion under an oscillator-like potential

Let us consider a physical system with a Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m} v^2 (\partial_u^2 + \partial_v^2) + v^2 (A + \frac{1}{2} m \omega^2 v^2) \quad (4.2.1)$$

for some constants A, ω . The potential is sometimes called oscillator like [9]. As in the free-particle case, we put

$$\hat{H} g(v) f(u) = E g(v) f(u) \quad (4.2.2)$$

and decompose it into two differential equations

$$\partial_u^2 f(u) = -C^2 f(u) \quad (4.2.3)$$

and

$$v^2 \partial_v^2 g(v) - (av^2 + bv^4 + \lambda)g(v) = 0 \quad (4.2.4)$$

where

$$a = C^2 + \frac{2m}{\hbar^2} A \quad b = \left(\frac{m\omega}{\hbar}\right)^2 \quad \lambda = -\frac{2m}{\hbar^2} E. \tag{4.2.5}$$

The differential equation for u in (4.2.3) has a solution given by equation (4.1.8) with C in (4.1.9). If we put $y = v^2$, then from equation (4.2.4) we obtain

$$4y^2 \frac{d^2g}{dy^2} + 2y \frac{dg}{dy} - (ay + by^2 + \lambda)g = 0. \tag{4.2.6}$$

Moreover, if we put $g = y^{-1/4}\phi(y)$, then we have

$$y^2 \frac{d^2\phi(y)}{dy^2} - \frac{1}{4}(ay + by^2 + \lambda - \frac{3}{4})\phi(y) = 0. \tag{4.2.7}$$

Now let us write $\phi(y)$ as the following form:

$$\phi(y) = y^\beta e^{-Ky} \xi(2Ky). \tag{4.2.8}$$

Then equation (4.2.7) becomes a Laguerre differential equation for $z = 2Ky$

$$z\xi''(z) + (2\beta - z)\xi'(z) + \left(-\frac{a}{8K} - \beta\right)\xi(z) = 0 \tag{4.2.9}$$

if β and $K(K > 0)$ satisfy

$$\frac{1}{4}(\lambda - \frac{3}{4}) = \beta(\beta - 1) \quad K^2 = \frac{1}{4}b. \tag{4.2.10}$$

Thus bound state solutions are obtained by the associated Laguerre polynomials

$$\xi(z) = L_n^{(\nu)}(z) \tag{4.2.11}$$

with

$$\nu + 1 = 2\beta > 0 \quad n = -\frac{a}{8K} - \beta \tag{4.2.12}$$

where A should be negative enough such that $a < 0$ for the existence of bound states. If we put $V_h = (\hbar^2/2m)a$, that is

$$V_h = -\frac{\hbar^2}{2m} \left[\frac{1 - e^{-2ikh}}{2h} \right]^2 + A \tag{4.2.13}$$

then we have up to constant multiplication

$$g(v) = v^{|V_h|/\hbar\omega - 2n - 1/2} \exp\left[-\frac{m\omega}{2\hbar} v^2\right] L_n^{(|V_h|/\hbar\omega - 2n - 1)}\left(\frac{m\omega}{\hbar} v^2\right) \tag{4.2.14}$$

with the energy eigenvalues

$$E_n = \frac{\hbar^2}{8m} - \frac{\hbar^2}{2m} \left(\frac{|V_h|}{\hbar\omega} - 2n - 1\right)^2 \tag{4.2.15}$$

for $n = 0, 1, \dots, N_M < |V_h|/2\hbar\omega$. The interpretation of this result in the commutative limit is made in [9]. On the h -deformed quantum plane, not only the energy eigenvalues but also the number of bound states depend explicitly on the deformation parameter h . In particular, we note that a cannot be less than 0 when $k \rightarrow \infty$ in the commutative limit and thus bound states cannot exist. However, for some h , $C^2 \rightarrow -1/4h^2$ when $k \rightarrow \infty$ and hence bound states can still exist whenever such h satisfies

$$A < \frac{\hbar^2}{8mh^2}. \tag{4.2.16}$$

4.3. Motion under a Coulomb-like potential

Now we consider a particle under a Coulomb-like potential [9] with a Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}v^2(\partial_u^2 + \partial_v^2) + v^2\left(A + \frac{B}{2m}v^{-1}\right) \quad (4.3.1)$$

for some constants A and B . As in the previous subsection, we decompose the Schrödinger equation into two differential equations

$$\partial_u^2 f(u) = -C^2 f(u) \quad (4.3.2)$$

and

$$v^2 \partial_v^2 g(v) - (av + bv^2 + \lambda)g(v) = 0 \quad (4.3.3)$$

where

$$a = \frac{B}{\hbar^2} \quad b = C^2 + \frac{2m}{\hbar^2}A \quad \lambda = -\frac{2m}{\hbar^2}E. \quad (4.3.4)$$

As before, we put

$$g(v) = v^\beta e^{-Kv} \xi(2Kv). \quad (4.3.5)$$

Then we have a Laguerre differential equation again for $z = 2Kv$

$$z\xi''(z) + (2\beta - z)\xi'(z) + \left(-\frac{a}{2K} - \beta\right)\xi(z) = 0 \quad (4.3.6)$$

if β and K ($K > 0$) satisfy

$$\lambda = \beta(\beta - 1) \quad K^2 = b. \quad (4.3.7)$$

Thus bound state solutions are also obtained by the associated Laguerre polynomials

$$\xi(z) = L_n^{(\nu)}(z) \quad (4.3.8)$$

with

$$v + 1 = 2\beta > 0 \quad n = -\frac{B}{2K\hbar^2} - \beta. \quad (4.3.9)$$

Here B should be less than 0 and A should be such that $K > 0$ for the existence of the bound states. If we put $V_h = (\hbar^2/2m)C^2 + A$, then we have up to constant multiplication

$$g(v) = v^{|B|/(2\hbar\sqrt{2mV_h})-n} \exp\left[-\frac{\sqrt{2mV_h}}{\hbar}v\right] L_n^{(|B|/(\hbar\sqrt{2mV_h})-2n-1)}\left(2\frac{\sqrt{2mV_h}}{\hbar}v\right) \quad (4.3.10)$$

with the energy eigenvalues

$$E_n = \frac{\hbar^2}{8m} - \frac{\hbar^2}{2m}\left(\frac{|B|}{2\hbar\sqrt{2mV_h}} - n - \frac{1}{2}\right)^2 \quad (4.3.11)$$

for $n = 0, 1, \dots, N_M < |B|/(2\hbar\sqrt{2mV_h})$. Not only the energy eigenvalues but also the number of bound states depend on the deformation parameter h as in the previous oscillator-like potential case. In particular, no bound states can exist in the commutative limit when $k \rightarrow \infty$ since $V_h \rightarrow \infty$. However, bound states can still exist for some h satisfying

$$A > \frac{\hbar^2}{8mh^2}. \quad (4.3.12)$$

5. Conclusions

On the h -deformed quantum plane, we have constructed the h -deformed Heisenberg algebra that is covariant under the quantum group $GL_h(2)$. It is worth comparing it with the deformed Heisenberg algebra constructed by Aghamohammadi [1], which is not covariant under the quantum group. We have also constructed the Laplace–Beltrami operator on the quantum plane. This operator has the Laplace–Beltrami operator of the Poincaré half-plane as its commutative limit.

We have introduced the Schrödinger equations for a free particle and particles under two specific potentials on the quantum plane and find their solutions explicitly by taking values in the noncommutative algebra. In the commutative limit the behaviour of a quantum particle on the quantum plane becomes that of the quantum particle on the Poincaré half-plane, a surface of constant negative Gaussian curvature. One usually expects that the energy spectra would depend on the parameter h explicitly at ‘large’ momentum and this is the case for the two specific potentials in this work. Moreover, it has been shown that bound states can survive on the quantum plane in a limiting case where bound states on the Poincaré half-plane disappear. The bound state solutions for the two potentials are alike since the corresponding Schrödinger equations can be reduced to the differential equation for the confluent hypergeometric functions.

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